LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – MATHEMATICS

THIRD SEMESTER – APRIL 2010

MT 3810 / 3803 - TOPOLOGY

Date & Time: 21/04/2010 / 9:00 - 12:00 Dept. No. Max. : 100 Marks

1) a) i) Let X be a non-empty set, and let d be a real function of ordered pairs of elements of X which satisfies the following two conditions:
$d(x,y)=0 \Leftrightarrow x=y, \text{ and } d(x,y) \le d(x,z)+d(y,z)$.
Show that d is a metric on X.

OR

ii) Let X be a metric space. Prove that any arbitrary union of open sets in X is open and any finite intersection of open sets in X is open. (5)

b) i) If a convergent sequence in a metric space has infinitely many distinct points, prove that its limit is a limit point of the set of points of the sequence.

ii) State and prove Cantor's Intersection Theorem.

iii) If $\{A_n\}$ is a sequence of nowhere dense sets in a complete metric space X, show that there exists a point in X which is not in any of the A_n's. (5+6+4)

OR

iv) Prove that the set $C(X,¡)$ of all bounded continuous real functions defined on a metric space X is a real Banach space with respect to pointwise addition and scalar multiplication and the norm defined by $\|f\| = \sup|f(x)|$.

2) a) i) If a metric space X is second countable, show that it is separable.

OR

ii) Define a topology on a non-empty set X with an example. Let X be a topological space and A be an arbitrary subset of X. Show that $\overline{A} = \{x /$ each neighbourhood of x intersects $A\}$. (5)

b) i) Show that any closed subspace of a compact space is compact.

ii) Give an example to show that a proper subspace of a compact space need not be closed.

iii) Prove that any continuous image of a compact space is compact. (6+3+6)

OR

iv) A topological space is compact, if every subbasic open cover has a finite subcover. - Prove (15)

3) a) i) State and prove Tychonoff's Theorem.

OR

ii) Show that a metric space is compact $\Leftrightarrow$ it is complete and totally bounded. (5)

b) i) Prove that in a sequentially compact space, every open cover has a Lebesque number.

ii) Show that every sequentially compact metric space is totally bounded. (10+5)

OR

iii) State and prove Ascoli's Theorem. (15)

4) a) i) Show that every subspace of a Hausdorff space is also a Hausdorff.

OR (5)

ii) Prove that every compact Hausdorff Space is normal. (5)

b) i) Let X be a T_1-space. Show that X is normal $\Leftrightarrow$ each neighbourhood of a closed set F contains the closure of some neighbourhood of F.

ii) State and prove Uryshon's lemma.

OR

iii) If X is a second countable normal space, show that there exists a homeomorphism f of X onto a subspace of R^{∞}. (15)

5) a) i) Prove that any continuous image of a connected space is connected

OR

ii) Show that the components of a totally disconnected space are its points. (5)

b) i) Let X be a topological space and A be a connected subspace of X. If B is a subspace of X such that $A \subseteq B \subseteq \bar{A}$, then show that B is connected.

ii) If X is an arbitrary topological space, then prove the following:
1) each point in X is contained in exactly one component of X;
2) each connected subspace of X is contained in a component of X;
3) a connected subspace of X which is both open and closed is a component of X.
(3+12)

OR

iii) State and prove the Weierstrass Approximation Theorem. (15)

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